

# Indefinite trans-Sasakian manifold with semi-symmetric metric connection

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## Abstract

The objective of the present paper is to study of indefinite trans-Sasakian manifold with a semi-symmetric metric connection. We have found the relations between curvature tensors, Ricci curvature tensors and scalar curvature of indefinite trans-Sasakian manifolds with semi-symmetric metric connection and with metric connection. Also, we have proved some results on quasi-projectively flat and  $\varphi$ -projectively flat manifolds with respect to semi-symmetric metric connection. It is shown that the manifold satisfying  $\tilde{R}.\tilde{S} = 0$  is an  $\eta$ -Einstein manifold if  $\alpha = 0$  and  $\beta = constant$ . It is also proved that the manifold satisfying  $\tilde{P}.\tilde{S} = 0$  is an  $\eta$ -Einstein manifold if  $\alpha = 0$  and  $\beta = constant$ . Finally, we have obtained the conditions for the manifold with semi-symmetric metric connection to be conformally flat and  $\xi$ -conformally flat.

2010 Mathematics Subject Classification. **53C25**. 53C50

Keywords. Semi-symmetric metric connection, Curvature tensor, Ricci-curvature tensor.

## 1 Introduction

The notion of semi-symmetric linear connection on a differentiable manifold was introduced by Friedman and Schouten in 1924 [8] and metric connection with a torsion on a Riemannain manifold was studied by Hayden in 1932 [9]. Semi-symmetric metric connection on a Riemannain manifold had been given by Yano in 1970 [17] and later studied by K.S.Amur and S.S.Pujar [11], C.S.Bagewadi [1], U. C. De et al [7], Sharafuddin and Hussain [10] and others. U.C.De [7] and C.S.Bagewadi *et al.* [2, 3, 14] previously provided some results on the conservativeness of Projective, Pseudo projective, Conformal, Concircular, Quasi conformal curvature tensors on K-contact, Kenmotsu and trans-Sasakian manifolds.

A new class of  $n$ -dimensional almost contact manifold namely trans-Sasakian manifold was introduced by J. A. Oubina in 1985 [15] and further study about the local structures of trans-Sasakian manifolds was carried by J.C.Marrero [13]. As a natural generalization of both Sasakian and Kenmotsu manifolds, the notion of trans-Sasakian manifolds, which are closely related to the locally conformal Kahler manifolds introduced by Oubiñá [15]. Trans-Sasskian manifold of type  $(0, 0)$ ,  $(\alpha, 0)$ , and  $(0, \beta)$  are, respectively, called the cosymplectic,  $\alpha$ -Sasakian, and  $\beta$ -Kenmotsu manifolds, with  $\alpha, \beta$  being scalar functions. In particular, if  $\alpha = 0, \beta = 1; \alpha = 1, \beta = 0$ ; then a trans-Sasakian manifold becomes Kenmotsu and Sasakian manifolds respectively. Hence trans-Saskian structures give a large class of generalized Quasi- Sasakian structures. The concept of indefinite Sasakian manifolds was introduced by A. Bejancu and K. L. Duggal [4] and further study was taken up by Xufend and Xiaoli[18], Rakesh Kumar et al. [12]. De and Sarkar [6].

In this present paper we shall introduce relation between metric connection and semi-symmetric metric connection in an  $n$ -dimensional indefinite trans Sasakian manifold  $M^n$ . Also, we have

proved some results on curvature tensors, Ricci curvature tensors, scalar curvatures, quasi projectively flat,  $\varphi$ -projectively flat,  $\bar{R}.\bar{S} = 0$ ,  $\bar{P}.\bar{S} = 0$ , Weyl conformally flat, Weyl  $\xi$ -conformally flat respectively in  $n$ -dimensional indefinite trans-Sasakian manifold  $M^n$  with a semi-symmetric metric connection.

## 2 Preliminaries

An  $n$ -dimensional smooth manifold  $(M^n, g)$  is said to be an indefinite almost contact metric manifold [5], if it admits a  $(1, 1)$  tensor field  $\varphi$ , a structure vector field  $\xi$ , a 1-form  $\eta$  and an indefinite metric  $g$  such that

$$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad (2.1)$$

$$\eta(\xi) = 1, \quad (2.2)$$

$$g(\xi, \xi) = \varepsilon, \quad (2.3)$$

$$\eta(X) = \varepsilon g(X, \xi), \quad (2.4)$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y), \quad (2.5)$$

for all vector fields  $X, Y$  on  $M^n$ , where  $\varepsilon$  is 1 or  $-1$  according as  $\xi$  is space like or time like vector field and rank  $\varphi$  is  $n - 1$ . If

$$d\eta(X, Y) = g(X, \varphi Y), \quad (2.6)$$

then  $M^n(\varphi, \xi, \eta, g)$  is called an indefinite contact metric manifold. An indefinite almost contact metric manifold is called an indefinite trans-Sasakian manifold if

$$(\nabla_X \varphi)Y = \alpha\{g(X, Y)\xi - \varepsilon\eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \varepsilon\eta(Y)\varphi X\}, \quad (2.7)$$

for any  $X, Y \in \Gamma(TM^n)$ , where  $\nabla$  is metric connection of indefinite metric  $g$ ,  $\alpha$  and  $\beta$  are smooth functions on  $M^n$ .

From equations (2.1), (2.2), (2.3), (2.4), (2.5) and (2.7), we have

$$\nabla_X \xi = \varepsilon\{-\alpha\varphi X + \beta(X - \eta(X)\xi)\}, \quad (2.8)$$

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta\{g(X, Y) - \varepsilon\eta(X)\eta(Y)\}. \quad (2.9)$$

Moreover, on such a indefinite trans-Sasakian manifold  $M^n$  of dimension  $n$  with structure  $M^n(\varphi, \xi, \eta, g)$  the following relations holds:

$$\begin{aligned} R(X, Y)\xi &= (\alpha^2 - \beta^2)\{\eta(Y)X - \eta(X)Y\} \\ &\quad + 2\alpha\beta\{\eta(Y)\varphi X - \eta(X)\varphi Y\} + \varepsilon\{(Y\alpha)\varphi X \\ &\quad - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y\}, \end{aligned} \quad (2.10)$$

$$\begin{aligned} R(\xi, Y)Z &= (\alpha^2 - \beta^2)\{\varepsilon g(Y, Z)\xi - \eta(Z)Y\} + 2\alpha\beta\{\varepsilon g(Y, \varphi Z)\xi \\ &\quad + \eta(Z)\varphi Y\} + \varepsilon(Z\alpha)\varphi Y + \varepsilon g(Y, \varphi Z)(grad\alpha) \\ &\quad - \varepsilon g(\varphi Y, \varphi Z)(grad\beta) + \varepsilon(Z\beta)\{Y - \eta(Z)\xi\}, \end{aligned} \quad (2.11)$$

$$S(Z, \xi) = \{(n - 1)(\alpha^2 - \beta^2) - \varepsilon(\xi\beta)\}\eta(Z) - \varepsilon(\varphi Z)\alpha - \varepsilon(n - 2)(Z\beta), \tag{2.12}$$

$$S(\xi, \xi) = (n - 1)(\alpha^2 - \beta^2) - \varepsilon(n - 1)(\xi\beta), \tag{2.13}$$

$$Q\xi = \varepsilon(n - 1)(\alpha^2 - \beta^2)\xi - (\xi\beta)\xi + \varepsilon\varphi(\text{grad}\alpha) - \varepsilon(n - 2)(\text{grad}\beta), \tag{2.14}$$

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z - \frac{1}{(n - 2)}[S(Y, Z)X - S(X, Z)Y \\ &\quad + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{(n - 1)(n - 2)} \\ &\quad [g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{2.15}$$

$$\begin{aligned} g(R(\xi, Y)Z, \xi) &= [(\alpha^2 - \beta^2) - \varepsilon(\xi\beta)]g(Y, Z) \\ &\quad + [(\xi\beta) - \varepsilon(\alpha^2 - \beta^2)]\eta(Y)\eta(Z) \\ &\quad - [2\alpha\beta + \varepsilon(\xi\alpha)]g(\varphi Y, Z), \end{aligned} \tag{2.16}$$

$$r = S(e_i, e_i) = \sum_{i=1}^n \varepsilon_i {}'R(e_i, e_i, e_i, e_i). \tag{2.17}$$

### 3 Semi-symmetric metric connection

A affine connection  $\bar{\nabla}$  in  $M^n$  is called semi-symmetric connection [10], if its torsion tensor

$$\bar{T}(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y], \tag{3.1}$$

satisfies

$$\bar{T}(X, Y) = \eta(X)Y - \eta(Y)X. \tag{3.2}$$

Moreover, a semi-symmetric connection is called semi-symmetric metric connection if

$$(\bar{\nabla}_X g)(Y, Z) = 0. \tag{3.3}$$

If  $\nabla$  is metric connection and  $\bar{\nabla}$  is the semi-symmetric metric with non-vanishing torsion tensor  $T$  in  $M^n$ , then we have

$$T(X, Y) = \eta(Y)X - \eta(X)Y, \tag{3.4}$$

$$\bar{\nabla}_X Y - \nabla_X Y = \frac{1}{2}[T(X, Y) + T^|(X, Y) + T^|(X, Y)], \tag{3.5}$$

where

$$g(T(Z, X), Y) = g(T^|(X, Y), Z). \tag{3.6}$$

By using equations (2.4), (3.4) and (3.6), we get

$$\begin{aligned} g(T^|(X, Y), Z) &= g(\eta(X)Z - \eta(Z)X, Y), \\ g(T^|(X, Y), Z) &= \eta(X)g(Z, Y) - \varepsilon g(X, Y)g(\xi, Z), \\ T^|(X, Y) &= \eta(X)Y - \varepsilon g(X, Y)\xi, \end{aligned} \tag{3.7}$$

$$T^l(Y, X) = \eta(Y)X - \varepsilon g(X, Y)\xi, \quad (3.8)$$

From equations (3.4), (3.5), (3.7) and (3.8), we get

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - \varepsilon g(X, Y)\xi.$$

Let  $M^n$  be an  $n$ -dimensional indefinite trans-Sasakian manifold and  $\nabla$  be the metric connection on  $M^n$ . The relation between the semi-symmetric metric connection  $\bar{\nabla}$  and the metric connection  $\nabla$  on  $M^n$  is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - \varepsilon g(X, Y)\xi. \quad (3.9)$$

#### 4 Curvature tensor on indefinite trans-Sasakian manifold with semi-symmetric metric connection

Let  $M^n$  be an  $n$ -dimensional indefinite trans-Sasakian manifold. The curvature tensor  $\bar{R}$  of  $M^n$  with respect to the semi-symmetric metric connection  $\bar{\nabla}$  is defined by

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \nabla_{[X, Y]} Z. \quad (4.1)$$

By using equations (2.2), (2.4), (3.9) and (4.1), we get

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + (2\beta + \varepsilon)[g(X, Z)Y - g(Y, Z)X] \\ &\quad + (\beta + \varepsilon)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\xi \\ &\quad + (1 + \beta\varepsilon)\eta(Z)[\eta(Y)X - \eta(X)Y] \\ &\quad - \alpha[g(\varphi X, Z)Y - g(\varphi Y, Z)X - g(Y, Z)\varphi X + g(X, Z)\varphi Y], \end{aligned} \quad (4.2)$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

is the Riemannian curvature tensor of metric connection  $\nabla$ .

Taking inner product of equation (4.2) with  $U$  and using equation (2.4), we have

$$\begin{aligned} &{}'\bar{R}(X, Y, Z, U) \\ &= {}'R(X, Y, Z, U) + (\varepsilon + 2\beta)[g(X, Z)g(Y, U) - g(Y, Z)g(X, U)] \\ &\quad + (1 + \varepsilon\beta)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\eta(U) \\ &\quad + (1 + \varepsilon\beta)[\eta(Y)g(X, U) - \eta(X)g(Y, U)]\eta(Z) \\ &\quad - \alpha[g(\varphi X, Z)g(Y, U) - g(\varphi Y, Z)g(X, U) - g(Y, Z)g(\varphi X, U) \\ &\quad + g(X, Z)g(\varphi Y, U)], \end{aligned} \quad (4.3)$$

where  $'\bar{R}(X, Y, Z, U) = g(\bar{R}(X, Y)Z, U)$  and  $'R(X, Y, Z, U) = g(R(X, Y)Z, U)$ .

Let  $\{e_1, e_2, \dots, e_{n-1}, \xi\}$  be a local orthonormal basis of vector fields on indefinite trans-Sasakian manifold  $M^n$ , then  $\{\varphi e_1, \varphi e_2, \dots, \varphi e_{n-1}, \xi\}$  is a also local orthonormal basis of vector

fields on indefinite trans-Sasakian manifold  $M^n$ . Now, putting  $X = U = e_i$  in the equation (4.3) and taking summation over  $i, 1 \leq i \leq n - 1$ , we get

$$\begin{aligned}
 & \sum_{i=1}^{n-1} {}' \bar{R}(e_i, Y, Z, e_i) \tag{4.4} \\
 = & \sum_{i=1}^{n-1} {}' R(e_i, Y, Z, e_i) + (2\beta + \varepsilon) \sum_{i=1}^{n-1} [g(e_i, Z)g(Y, e_i) - g(Y, Z)g(e_i, e_i)] \\
 & + (\beta + \varepsilon) \sum_{i=1}^{n-1} [\varepsilon g(Y, Z)g(e_i, \xi)g(\xi, e_i) - \eta(Y)g(e_i, Z)g(\xi, e_i)] \\
 & + (\beta\varepsilon + 1) \sum_{i=1}^{n-1} [\eta(Z)\eta(Y)g(e_i, e_i) - \varepsilon\eta(Z)g(e_i, \xi)g(Y, e_i)] \\
 & - \alpha \sum_{i=1}^{n-1} [g(e_i\varphi, Z)g(Y, e_i) - g(\varphi Y, Z)g(e_i, e_i) - g(Y, Z)g(\varphi e_i, e_i) \\
 & + g(e_i, Z)g(\varphi Y, e_i)].
 \end{aligned}$$

We have using

$$S(Y, Z) = \sum_{i=1}^n \varepsilon_i {}' R(e_i, Y, Z, e_i), \quad g(Y, Z) = \sum_{i=1}^n \varepsilon_i g(Y, e_i)g(e_i, Z), \tag{4.5}$$

$$\bar{S}(Y, Z) = \sum_{i=1}^n \varepsilon_i {}' \bar{R}(e_i, Y, Z, e_i). \tag{4.6}$$

Also

$$\sum_{i=1}^{n-1} {}' \bar{R}(e_i, Y, Z, e_i) = \bar{S}(Y, Z) - \varepsilon {}' \bar{R}(\xi, Y, Z, \xi), \tag{4.7}$$

$$\sum_{i=1}^{n-1} {}' R(e_i, Y, Z, e_i) = S(Y, Z) - \varepsilon {}' R(\xi, Y, Z, \xi), \tag{4.8}$$

$$\sum_{i=1}^{n-1} g(e_i, Z)g(Y, e_i) = g(Y, Z) - \varepsilon g(\xi, Z)g(Y, \xi), \tag{4.9}$$

$$\sum_{i=1}^{n-1} g(e_i, e_i) = n - 1, \tag{4.10}$$

$$\sum_{i=1}^{n-1} g(e_i, \xi)g(\xi, e_i) = 0, \tag{4.11}$$

$$\sum_{i=1}^{n-1} g(e_i, Z)g(\xi, e_i) = 0, \tag{4.12}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, Z)g(Y, e_i) = g(\varphi Y, Z), \tag{4.13}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, e_i) = 0, \tag{4.14}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = n - 1, \tag{4.15}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, Z)g(Y, \varphi e_i) = g(Y, Z) - \varepsilon g(\xi, Z)g(Y, \xi). \tag{4.16}$$

Hence, by virtue of equations (4.4) and (4.7) – (4.16), we get

$$\begin{aligned} \bar{S}(Y, Z) &= S(Y, Z) - [(2\beta + \varepsilon)(n - 2) + \beta]g(Y, Z) \\ &\quad + (1 + \varepsilon\beta)(n - 2)\eta(Z)\eta(Y) + \alpha(n - 2)g(\varphi Y, Z), \end{aligned} \tag{4.17}$$

where  $\bar{S}$  and  $S$  are the Ricci tensor of connection  $\bar{\nabla}$  and  $\nabla$ , respectively in  $M^n$ .

$$\bar{Q}Y = QY - [(2\beta + \varepsilon)(n - 2) + \beta]Y + (\varepsilon + \beta)(n - 2)\eta(Y)\xi + \alpha(n - 2)\varphi Y, \tag{4.18}$$

where  $\bar{Q}$  and  $Q$  are Ricci operator with respect to the semi-symmetric metric connection and metric connection respectively and are define as  $g(\bar{Q}Y, Z) = \bar{S}(Y, Z)$  and  $g(QY, Z) = S(Y, Z)$  respectively.

Replace  $Y = \xi$  in (4.18) and using (2.14), we get

$$\begin{aligned} \bar{Q}\xi &= \varepsilon(n - 1)(\alpha^2 - \beta^2)\xi - (\xi\beta)\xi + \varepsilon\varphi(\text{grad}\alpha) \\ &\quad - \varepsilon(n - 2)(\text{grad}\beta) - \beta(n - 1)\xi. \end{aligned} \tag{4.19}$$

Putting  $Y = Z = e_i$  and taking summation over  $i, 1 \leq i \leq n - 1$  in equation (4.17), using equations (2.13), (2.17), (4.10), (4.11) and (4.14), we get

$$\bar{r} = r - (n - 1)[(2\beta + \varepsilon)(n - 2) + 2\beta], \tag{4.20}$$

where  $\bar{r}$  and  $r$  are the scalar curvatures of the connection  $\bar{\nabla}$  and  $\nabla$ , respectively.

Replace  $Y \rightarrow \varphi Y$  in equation (4.17), we get

$$\begin{aligned} \bar{S}(\varphi Y, Z) &= S(\varphi Y, Z) - [(2\beta + \varepsilon)(n - 2) + \beta]g(\varphi Y, Z) \\ &\quad - \alpha(n - 2)g(Y, Z) + \varepsilon\alpha(n - 2)\eta(Z)\eta(Y). \end{aligned} \tag{4.21}$$

Using equations (2.1), (2.2), (2.4) and (2.12) in equation (4.21), we get

$$\bar{S}(\varphi Y, \xi) = S(\varphi Y, \xi) = -\varepsilon(\varphi^2 Y)\alpha - \varepsilon(n - 2)(\varphi Y)\beta. \tag{4.22}$$

Using equations (2.1), (2.2), (2.4), (2.12), (2.13) and (4.17), we get

$$\begin{aligned} \bar{S}(Y, \xi) &= [(n - 1)(\alpha^2 - \beta^2) - \varepsilon(\xi\beta) - \varepsilon\beta(n - 1)]\eta(Y) \\ &\quad - \varepsilon(\varphi Y)\alpha - \varepsilon(n - 2)(Y\beta), \end{aligned} \tag{4.23}$$

$$\bar{S}(\xi, \xi) = (n - 1)(\alpha^2 - \beta^2) - \varepsilon\beta(n - 1) - \varepsilon(n - 1)(\xi\beta), \tag{4.24}$$

$$\begin{aligned} \bar{S}(grad\alpha, \xi) &= \varepsilon(n - 1)(\alpha^2 - \beta^2)(\xi\alpha) - \beta(n - 1)(\xi\alpha) - (\xi\alpha)(\xi\beta) \\ &\quad - \varepsilon(\varphi grad\alpha)\alpha - \varepsilon(n - 2)g(grad\alpha, grad\beta), \end{aligned} \tag{4.25}$$

$$\begin{aligned} \bar{S}(grad\beta, \xi) &= \varepsilon(n - 1)(\alpha^2 - \beta^2)(\xi\beta) - \beta(n - 1)(\xi\beta) - (\xi\beta)^2 \\ &\quad - \varepsilon(\varphi grad\beta)\alpha - \varepsilon(n - 2)(grad\beta)^2. \end{aligned} \tag{4.26}$$

From equation (4.2) by the cyclic permutations of  $X, Y$  and  $Z$ , we get

$$\begin{aligned} &\bar{R}(Y, Z)X \\ &= R(Y, Z)X + (\varepsilon + 2\beta)[g(Y, X)Z - g(Z, X)Y] \\ &\quad + (\varepsilon + \beta)[\eta(Y)g(Z, X) - \eta(Z)g(Y, X)]\xi \\ &\quad + (1 + \beta\varepsilon)\eta(X)[\eta(Z)Y - \eta(Y)Z] \\ &\quad - \alpha[g(\varphi Y, X)Z - g(\varphi Z, X)Y - g(Z, X)\varphi Y + g(Y, X)\varphi Z], \end{aligned} \tag{4.27}$$

$$\begin{aligned} &\bar{R}(Z, X)Y \\ &= R(Z, X)Y + (2\beta + \varepsilon)[g(Z, Y)X - g(X, Y)Z] \\ &\quad + (\varepsilon + \beta)[\eta(Y)[\eta(Z)g(X, Y) - \eta(X)g(Y, Z)]\xi \\ &\quad + (1 + \varepsilon\beta)\eta(Y)[\eta(X)Z - \eta(Z)X] \\ &\quad - \alpha[g(\varphi Z, Y)X - g(\varphi X, Y)Z - g(X, Y)\varphi Z + g(Z, Y)\varphi X]. \end{aligned} \tag{4.28}$$

Adding equations (4.2), (4.27) and (4.28), using the Bianchi' first identity, we get

$$\begin{aligned} &\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y \\ &= 2\alpha[g(\varphi Y, Z)X + g(\varphi Z, X)Y + g(\varphi X, Y)Z]. \end{aligned} \tag{4.29}$$

If  $\alpha = 0$  in equation (4.29), we have

$$\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0.$$

**Theorem 4.1.** Let  $M^n$  be an  $n$ -dimensional indefinite trans-Sasakian manifold with semi-symmetric metric connection  $\bar{\nabla}$  and curvature tensor  $\bar{R}$  satisfies Bianchi first identity if and only if  $\alpha = 0$ , i.e.  $M^n$  is a  $\beta$ -Kenmotsu manifold.

Now, from equation (4.3) interchanging  $X$  and  $Y$ , we get

$$\begin{aligned} &{}'\bar{R}(Y, X, Z, U) \\ &= {}'R(Y, X, Z, U) + (\varepsilon + 2\beta)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)] \\ &\quad + (1 + \varepsilon\beta)[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)]\eta(U) \\ &\quad + (1 + \varepsilon\beta)[\eta(X)g(Y, U) - \eta(Y)g(X, U)]\eta(Z) \\ &\quad - \alpha[g(\varphi Y, Z)g(X, U) - g(\varphi X, Z)g(Y, U) - g(X, Z)g(\varphi Y, U) \\ &\quad + g(Y, Z)g(\varphi X, U)]. \end{aligned} \tag{4.30}$$

From equations (4.3) and (4.30), we get

$${}'\bar{R}(X, Y, Z, U) = -{}'\bar{R}(Y, X, Z, U), \quad (4.31)$$

where

$${}'R(X, Y, Z, U) = -{}'R(Y, X, Z, U).$$

Again from equation (4.3) interchanging  $Z$  and  $U$ , we get

$$\begin{aligned} & {}'\bar{R}(X, Y, U, Z) \\ = & {}'R(X, Y, U, Z) + (2\beta + \varepsilon)[g(X, U)g(Y, Z) - g(Y, U)g(X, Z)] \\ & + (1 + \varepsilon\beta)[\eta(X)g(Y, U) - \eta(Y)g(X, U)]\eta(Z) \\ & + (1 + \varepsilon\beta)[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)]\eta(U) \\ & - \alpha[g(\varphi X, U)g(Y, Z) - g(\varphi Y, U)g(X, Z) - g(Y, U)g(\varphi X, Z) \\ & + g(X, U)g(\varphi Y, Z)]. \end{aligned} \quad (4.32)$$

From equations (4.3) and (4.32), we get

$${}'\bar{R}(X, Y, Z, U) = -{}'\bar{R}(X, Y, U, Z), \quad (4.33)$$

where

$${}'R(X, Y, Z, U) = -{}'R(X, Y, U, Z).$$

Again from equation (4.3) interchanging pair of slots, we get

$$\begin{aligned} & {}'\bar{R}(Z, U, X, Y) \\ = & {}'R(Z, U, X, Y) + (2\beta + \varepsilon)[g(Z, X)g(U, Y) - g(U, X)g(Z, Y)] \\ & + (1 + \varepsilon\beta)[\eta(Z)g(U, X) - \eta(U)g(Z, X)]\eta(Y) \\ & + (1 + \varepsilon\beta)[\eta(U)g(Z, Y) - \eta(Z)g(U, Y)]\eta(X) \\ & - \alpha[g(\varphi Z, X)g(U, Y) - g(\varphi U, X)g(Z, Y) - g(\varphi Z, Y)g(U, X) \\ & + g(\varphi U, Y)g(Z, X)]. \end{aligned} \quad (4.34)$$

From equations (4.3) and (4.34), we get

$$\begin{aligned} & {}'\bar{R}(X, Y, Z, U) \\ = & {}'\bar{R}(Z, U, X, Y) + 2\alpha[g(\varphi Z, X)g(U, Y) \\ & + g(\varphi Y, Z)g(X, U) + g(\varphi X, U)g(Y, Z) \\ & + g(\varphi U, Y)g(Z, X)]. \end{aligned} \quad (4.35)$$

If  $\alpha = 0$  in equation (4.35), we have

$${}'\bar{R}(X, Y, Z, U) = {}'\bar{R}(Z, U, X, Y)$$

**Theorem 4.2.** The curvature tensor  $\bar{R}$  of type (0, 4) of semi-symmetric metric connection  $\bar{\nabla}$  is an indefinite trans-Sasakian manifolds is



- (i) Skew symmetric in first two slots.
- (ii) Skew symmetric in last two slots.
- (iii) Symmetric in pair of slots if and only if  $\alpha = 0$  in equation (4.35), then  $M^n$  is a  $\beta$ -Kenmotsu manifold.

Now, let  $\bar{R}(X, Y)Z = 0$  in equation (4.2), we get

$$\begin{aligned}
 R(X, Y)Z &= (\varepsilon + 2\beta)[g(Y, Z)X - g(X, Z)Y] \\
 &+ (\varepsilon + \beta)[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)]\xi \\
 &+ (1 + \varepsilon\beta)\eta(Z)[\eta(X)Y - \eta(Y)X] \\
 &+ \alpha[g(\varphi X, Z)Y - g(\varphi Y, Z)X - g(Y, Z)\varphi X \\
 &+ g(X, Z)\varphi Y],
 \end{aligned}
 \tag{4.36}$$

Taking the inner product of equation (4.36) with  $\xi$  and using (2.4), we get

$$\begin{aligned}
 \varepsilon\eta(R(X, Y)Z) &= \varepsilon\beta[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\
 &+ \varepsilon\alpha[g(\varphi X, Z)\eta(Y) - g(\varphi Y, Z)\eta(X)].
 \end{aligned}
 \tag{4.37}$$

Using equation (2.4) in equation (4.37), we get

$$R(X, Y)Z = \beta[g(Y, Z)X - g(X, Z)Y] + \alpha[g(\varphi X, Z)Y - g(\varphi Y, Z)X].
 \tag{4.38}$$

**Theorem 4.3.** If the curvature tensor  $\bar{R}$  of a semi-symmetric metric connection in an indefinite trans-Sasakian manifold  $M^n$  vanishes, then the indefinite trans sasakian manifold is of constant curvature if  $\alpha = 0$ . i.e.  $M^n$  is  $\beta$ -Kenmotsu manifold.

Now, in equation (4.2) putting  $Z = \xi$ , using equations (2.1), (2.2), (2.4) and (2.10), we get

$$\begin{aligned}
 \bar{R}(X, Y)\xi &= (\alpha^2 - \beta^2 - \varepsilon\beta)[\eta(Y)X - \eta(X)Y] \\
 &+ (2\alpha\beta + \varepsilon\alpha)[\eta(Y)\varphi X - \eta(X)\varphi Y] \\
 &+ \varepsilon[(Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y].
 \end{aligned}
 \tag{4.39}$$

Replace  $Y = \xi$  in equation (4.39), using equations (2.1), (2.2) and (2.4), we get

$$\begin{aligned}
 \bar{R}(X, \xi)\xi &= (\alpha^2 - \beta^2 - \varepsilon\beta)[X - \eta(X)\xi] \\
 &+ [2\alpha\beta + \varepsilon\alpha + \varepsilon(\xi\alpha)]\varphi X + \varepsilon(\xi\beta)\varphi^2 X.
 \end{aligned}
 \tag{4.40}$$

Now, again replace  $X = \xi$  in equation (4.39), using equations (2.1), (2.2) and (2.4), we get

$$\begin{aligned}
 \bar{R}(\xi, Y)\xi &= (\alpha^2 - \beta^2 - \varepsilon\beta)[\eta(Y)\xi - Y] - [2\alpha\beta + \varepsilon\alpha \\
 &+ \varepsilon(\xi\alpha)]\varphi Y - \varepsilon(\xi\beta)\varphi^2 Y.
 \end{aligned}
 \tag{4.41}$$

Replace  $Y = X$  in equation (4.41), we get

$$\begin{aligned}
 \bar{R}(\xi, X)\xi &= -(\alpha^2 - \beta^2 - \varepsilon\beta)[X - \eta(X)\xi] - [2\alpha\beta + \varepsilon\alpha \\
 &- \varepsilon(\xi\alpha)]\varphi X - \varepsilon(\xi\beta)\varphi^2 X.
 \end{aligned}
 \tag{4.42}$$

From equations (4.40) and (4.42), we get

$$\bar{R}(X, \xi)\xi = -\bar{R}(\xi, X)\xi.
 \tag{4.43}$$

**Theorem 4.4.** Let  $M^n$  be an  $n$ -dimensional indefinite trans-Sasakian manifold with a semi-symmetric metric connection, then

$$\begin{aligned}\bar{R}(X, Y)\xi &= (\alpha^2 - \beta^2 - \varepsilon\beta)[\eta(Y)X - \eta(X)Y] \\ &\quad + (2\alpha\beta + \varepsilon\alpha)[\eta(Y)\varphi X - \eta(X)\varphi Y] \\ &\quad + \varepsilon[(Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y].\end{aligned}\tag{4.44}$$

**Lemma 4.5.** Let  $M^n$  be an  $n$ -dimensional indefinite trans-Sasakian manifold with a semi-symmetric metric connection, then

$$(\bar{\nabla}_X \eta)(Y) = (\beta + \varepsilon)g(X, Y) - (1 + \varepsilon\beta)\eta(X)\eta(Y) - \alpha g(\varphi X, Y),\tag{4.45}$$

$$\bar{\nabla}_X \xi = (1 + \varepsilon\beta)X - (1 + \varepsilon\beta)\eta(X)\xi - \varepsilon\alpha(\varphi X).\tag{4.46}$$

*Proof.* By the covariant differentiation of  $\eta(Y)$  with respect  $X$ , we have

$$\begin{aligned}\bar{\nabla}_X \eta(Y) &= (\bar{\nabla}_X \eta)(Y) + \eta(\bar{\nabla}_X Y), \\ (\bar{\nabla}_X \eta)(Y) &= \bar{\nabla}_X \eta(Y) - \eta(\bar{\nabla}_X Y).\end{aligned}$$

By using equations (2.4) and (3.3), we get

$$(\bar{\nabla}_X \eta)(Y) = \varepsilon g(Y, \bar{\nabla}_X \xi).\tag{4.47}$$

On putting  $Y = \xi$  in equation (3.9), we have

$$\bar{\nabla}_X \xi = \nabla_X \xi + \eta(\xi)X - \varepsilon g(X, \xi)\xi.$$

By using equation (2.4), we get

$$\bar{\nabla}_X \xi = \nabla_X \xi + X - \eta(X)\xi,\tag{4.48}$$

From equation (4.47), using equations (2.4), (2.8) and (4.48), we get

$$(\bar{\nabla}_X \eta)(Y) = (\beta + \varepsilon)g(X, Y) - (1 + \varepsilon\beta)\eta(X)\eta(Y) - \alpha g(\varphi X, Y).$$

From equations (2.8) and (4.6), we get

$$\bar{\nabla}_X \xi = (1 + \varepsilon\beta)X - (1 + \varepsilon\beta)\eta(X)\xi - \varepsilon\alpha(\varphi X).$$

Q.E.D.

### 5 Quasi-projectively flat indefinite trans-Sasakian manifold with respect to semi-symmetric metric connection

Let  $M^n$  be an  $n$ -dimensional indefinite trans-Sasakian manifold. If there exists a one to one correspondence between each co-ordinate neighbourhood of  $M^n$  and a domain in Euclidean space such that any geodesic of the indefinite trans-Sasakian manifold corresponds to a straight line in the Euclidean space, then  $M^n$  is said to be locally projectively flat. The projective curvature tensor  $\bar{P}$  with respect to semi-symmetric metric connection is defined by

$$\bar{P}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{(n-1)}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y]. \tag{5.1}$$

**Definition 5.1.** An indefinite trans-Sasakian manifold  $M^n$  is said to be quasi-projectively flat with respect to semi-symmetric metric connection, if

$$g(\bar{P}(\varphi X, Y)Z, \varphi U) = 0, \tag{5.2}$$

where  $\bar{P}$  is the projective curvature tensor with respect to semi-symmetric metric connection. From equation (5.1) taking inner product with  $U$ , we get

$$g(\bar{P}(X, Y)Z, U) = g(\bar{R}(X, Y)Z, U) - \frac{1}{(n-1)}[\bar{S}(Y, Z)g(X, U) - \bar{S}(X, Z)g(Y, U)]. \tag{5.3}$$

Replace  $X = \varphi X$  and  $U = \varphi U$  in equation (5.3), we get

$$g(\bar{P}(\varphi X, Y)Z, \varphi U) = g(\bar{R}(\varphi X, Y)Z, \varphi U) - \frac{1}{(n-1)}[\bar{S}(Y, Z)g(\varphi X, \varphi U) - \bar{S}(\varphi X, Z)g(Y, \varphi U)]. \tag{5.4}$$

From equations (5.2) and (5.4), we have

$$g(\bar{R}(\varphi X, Y)Z, \varphi U) = \frac{1}{(n-1)}[\bar{S}(Y, Z)g(\varphi X, \varphi U) - \bar{S}(\varphi X, Z)g(Y, \varphi U)]. \tag{5.5}$$

Now, using equations (2, 1), (2.4), (4.17) and (4.21) in equation (5.5), we have

$$\begin{aligned} &g(R(\varphi X, Y)Z, \varphi U) \tag{5.6} \\ = &\frac{1}{(n-1)}[S(Y, Z)g(\varphi X, \varphi U) - S(\varphi X, Z)g(Y, \varphi U)] \\ &- \frac{(\beta + \varepsilon)}{(n-1)}g(\varphi X, Z)g(Y, \varphi U) + \frac{(\beta + \varepsilon)}{(n-1)}g(Y, Z)g(\varphi X, \varphi U) \\ &- \frac{(1 + \varepsilon\beta)}{(n-1)}\eta(Y)\eta(Z)g(\varphi X, \varphi U) + \frac{\varepsilon\alpha}{(n-1)}\eta(X)\eta(Z)g(Y, \varphi U) \\ &- \frac{\alpha}{(n-1)}g(X, Z)g(Y, \varphi U) - \frac{\alpha}{(n-1)}g(\varphi Y, Z)g(\varphi X, \varphi U) \\ &+ \alpha g(Y, Z)g(X, \varphi U) + \alpha g(\varphi X, Z)g(\varphi Y, \varphi U). \end{aligned}$$

Let  $\{e_1, e_2, \dots, e_{n-1}, \xi\}$  be a local orthonormal basis of vector fields on indefinite trans-Sasakian manifold  $M^n$ , then  $\{\varphi e_1, \varphi e_2, \dots, \varphi e_{n-1}, \xi\}$  is also local orthonormal basis of vector fields on indefinite trans-Sasakian manifold  $M^n$ . Now, putting  $X = U = e_i$  in equation (5.6) and taking summation over  $i, 1 \leq i \leq n - 1$ , we have

$$\begin{aligned} & \sum_{i=1}^{n-1} g(R(\varphi e_i, Y)Z, \varphi e_i) \tag{5.7} \\ = & \frac{1}{(n-1)} \sum_{i=1}^{n-1} [S(Y, Z)g(\varphi e_i, \varphi e_i) - S(\varphi e_i, Z)g(Y, \varphi e_i)] \\ & - \frac{(\beta + \varepsilon)}{(n-1)} \sum_{i=1}^{n-1} g(\varphi e_i, Z)g(Y, \varphi e_i) + \frac{(\beta + \varepsilon)}{(n-1)} \sum_{i=1}^{n-1} g(Y, Z)g(\varphi e_i, \varphi e_i) \\ & - \frac{(1 + \varepsilon\beta)}{(n-1)} \sum_{i=1}^{n-1} \eta(Y)\eta(Z)g(\varphi e_i, \varphi e_i) + \frac{\varepsilon\alpha}{(n-1)} \sum_{i=1}^{n-1} \eta(e_i)\eta(Z)g(Y, \varphi e_i) \\ & - \frac{\alpha}{(n-1)} \sum_{i=1}^{n-1} g(e_i, Z)g(Y, \varphi e_i) - \frac{\alpha}{(n-1)} \sum_{i=1}^{n-1} g(\varphi Y, Z)g(\varphi e_i, \varphi e_i) \\ & + \alpha \sum_{i=1}^{n-1} g(Y, Z)g(e_i, \varphi e_i) + \alpha \sum_{i=1}^{n-1} g(\varphi e_i, Z)g(\varphi Y, \varphi e_i). \end{aligned}$$

We have also

$$\sum_{i=1}^{n-1} S(\varphi e_i, Z)g(Y, \varphi e_i) = S(Y, Z) - \varepsilon g(\xi, Z)S(Y, \xi). \tag{5.8}$$

Now, using equations (2.1), (2.2), (2.4), (2.16), (4.8) – (4.16) and (5.8) in equation (5.7), we have

$$\begin{aligned} S(Y, Z) = & [(n-2)(\varepsilon + \beta) + \varepsilon(n-1)(\alpha^2 - \beta^2) - (n-1)(\xi\beta)]g(Y, Z) \tag{5.9} \\ & + [\varepsilon(n-2)(\xi\beta) - (n-2)(1 + \varepsilon\beta)]\eta(Y)\eta(Z) \\ & - [2\varepsilon(n-1)\alpha\beta + (n-1)(\xi\alpha) - \alpha]g(\varphi Y, Z) \\ & - \varepsilon\eta(Y)(\varphi Z)\alpha - \varepsilon(n-2)(\xi\beta)\eta(Y). \end{aligned}$$

If  $\alpha = 0$  and  $\beta = \text{constant}$  in equation (5.9), we get

$$S(Y, Z) = [(n-2)(\varepsilon + \beta) - (n-1)\varepsilon\beta^2]g(Y, Z) + (2-n)(1 + \varepsilon\beta)\eta(Y)\eta(Z). \tag{5.10}$$

Therefore,

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z),$$

where  $a = (n-2)(\varepsilon + \beta) - (n-1)\varepsilon\beta^2$  and  $b = (2-n)(1 + \varepsilon\beta)$ .

This result shows that the manifold under the consideration is an  $\eta$ -Einstein manifold. Thus, we can state the following theorem:

**Theorem 5.2.** An  $n$ -dimensional quasi projectively flat indefinite trans-Sasakin manifold  $M^n$  with respect to a semi-symmetric metric connection is  $\eta$ -Einstein manifold if  $\alpha = 0$  and  $\beta = \text{constant}$ .

### 6 $\varphi$ -Projectively flat indefinite trans Sasakian manifold with respect to a semi-symmetric metric connection

An  $M^n$  indefinite trans-Sasakian manifold with respect to a semi-symmetric metric connection is said be  $\varphi$ -projectively flat if

$$\varphi^2(\bar{P}(\varphi X, \varphi Y)\varphi Z) = 0, \tag{6.1}$$

where  $\bar{P}$  is the projective curvature tensor of the  $M^n$  indefinite trans-Sasakian manifold with respect to a semi-symmetric metric connection. Suppose  $M^n$  be a  $\varphi$ -projectively flat indefinite trans-Sasakian manifold with respect to a semi-symmetric metric connection . It is known that  $\varphi^2(\bar{P}(\varphi X, \varphi Y)\varphi Z) = 0$  holds if and only if

$$g(\bar{P}(\varphi X, \varphi Y)\varphi Z), \varphi U) = 0, \tag{6.2}$$

for any  $X, Y, Z, U \in TM^n$ . Replace  $Y = \varphi Y$  and  $U = \varphi U$  in the equation (5.4), we have

$$\begin{aligned} g(\bar{P}(\varphi X, Y\varphi)\varphi Z, \varphi U) &= g(\bar{R}(\varphi X, \varphi Y)\varphi Z, \varphi U) - \frac{1}{(n-1)} \\ &[\bar{S}(\varphi Y, \varphi Z)g(\varphi X, \varphi U) - \bar{S}(\varphi X, \varphi Z)g(\varphi Y, \varphi U)]. \end{aligned} \tag{6.3}$$

From equations (6.2) and (6.3), we have

$$\begin{aligned} g(\bar{R}(\varphi X, \varphi Y)\varphi Z, \varphi U) &= \frac{1}{(n-1)}[\bar{S}(\varphi Y, Z)g(\varphi X, \varphi U) \\ &- \bar{S}(\varphi X, \varphi Z)g(\varphi Y, \varphi U)]. \end{aligned} \tag{6.4}$$

Using equations (2.1), (2.2), (2.4), (2.5), (4.2) and (4.17) in equation (6.4), we have

$$\begin{aligned} &g(R(\varphi X, \varphi Y)\varphi Z, \varphi U) \tag{6.5} \\ &= \frac{1}{(n-1)}[S(\varphi Y, \varphi Z)g(\varphi X, \varphi U) - S(\varphi X, \varphi Z)g(\varphi Y, \varphi U)] \\ &+ \frac{(\varepsilon + \beta)}{(n-1)}g(\varphi Y, \varphi Z)g(\varphi X, \varphi U) - \frac{(\varepsilon + \beta)}{(n-1)}g(\varphi X, \varphi Z)g(\varphi Y, \varphi U) \\ &+ \frac{\alpha}{(n-1)}g(Y, \varphi Z)g(\varphi X, \varphi U) - \frac{\alpha}{(n-1)}g(X, \varphi Z)g(\varphi Y, \varphi U) \\ &+ \alpha g(\varphi Y, \varphi Z)g(X, \varphi U) - \alpha g(\varphi X, \varphi Z)g(Y, \varphi U). \end{aligned}$$

Let  $\{e_1, e_2, \dots, e_{n-1}, \xi\}$  be a local orthonormal basis of vector fields on indefinite trans-Sasakian manifold  $M^n$ , then  $\{\varphi e_1, \varphi e_2, \dots, \varphi e_{n-1}, \xi\}$  is a also local orthonormal basis of vector fields on indefinite trans-Sasakian manifold  $M^n$ . Now, replace  $X = U = e_i$  in equation (6.5) and taking

summation over  $i, 1 \leq i \leq n - 1$ , we get

$$\begin{aligned} & \sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) \tag{6.6} \\ = & \frac{1}{(n-1)} \sum_{i=1}^{n-1} [S(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i) - S(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i)] \\ & + \frac{(\varepsilon + \beta)}{(n-1)} \sum_{i=1}^{n-1} g(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i) - \frac{(\varepsilon + \beta)}{(n-1)} \sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i) \\ & + \frac{\alpha}{(n-1)} \sum_{i=1}^{n-1} g(Y, \varphi Z)g(\varphi e_i, \varphi e_i) - \frac{\alpha}{(n-1)} \sum_{i=1}^{n-1} g(e_i, \varphi Z)g(\varphi Y, \varphi e_i) \\ & + \alpha \sum_{i=1}^{n-1} g(\varphi Y, \varphi Z)g(e_i, \varphi e_i) - \alpha \sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z)g(Y, \varphi e_i) \end{aligned}$$

Now, using equations (2.1), (2.2), (2.4), (2.5), (2.16), (4.8) – (4.16) and (5.8) in (6.6), we have

$$\begin{aligned} & S(Y, Z) \tag{6.7} \\ = & [(n-2)(\varepsilon + \beta) + \varepsilon(n-1)(\alpha^2 - \beta^2) - (n-1)(\xi\beta)]g(Y, Z) \\ & + [2\varepsilon(n-2)(\xi\beta) - (n-2)(1 + \varepsilon\beta)]\eta(Y)\eta(Z) \\ & + [\alpha - 2\varepsilon\alpha\beta(n-1) - (n-1)(\xi\alpha)]g(\varphi Y, Z) \\ & - [\varepsilon(\varphi Z)\alpha + \varepsilon(n-2)(Z\beta)]\eta(Y) - [\varepsilon(\varphi Y)\alpha + \varepsilon(n-2)(Y\beta)]\eta(Z), \end{aligned}$$

If  $\alpha = 0$  and  $\beta = \text{constant}$  in equation (6.7), we get

$$S(Y, Z) = [(n-2)(\varepsilon + \beta) - (n-1)\varepsilon\beta^2]g(Y, Z) + (2-n)(1 + \varepsilon\beta)\eta(Y)\eta(Z). \tag{6.8}$$

Therefore

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z),$$

where  $a = (n-2)(\varepsilon + \beta) - (n-1)\varepsilon\beta^2$  and  $b = (2-n)(1 + \varepsilon\beta)$ .

This result shows that the manifold under the consideration is an  $\eta$ -Einstein manifold. Thus, we can state the following theorem:

**Theorem 6.1.** An  $n$ -dimensional  $\varphi$ -projectively flat indefinite trans-Sasakin manifold  $M^n$  with semi-symmetric metric connection is  $\eta$ -Einstein manifold if  $\alpha = 0$  and  $\beta = \text{constant}$ .

### 7 Indefinite trans-Sasakian manifold with a semi-symmetric metric connection satisfying $\bar{R}.\bar{S} = 0$

Now, suppose that an  $M^n$  be  $n$ -dimensional indefinite trans-Sasakian manifold with a semi-symmetric metric connection  $\bar{\nabla}$  satisfying the condition

$$\bar{R}(X, Y).\bar{S} = 0. \tag{7.1}$$

Then, we have

$$\bar{S}(\bar{R}(X, Y)Z, U) + \bar{S}(Z, \bar{R}(X, Y)U) = 0 \quad (7.2)$$

Now, replace  $X = \xi$  in equation (7.2), using equations (2.11) and (4.2), we have

$$\begin{aligned} & \varepsilon(\alpha^2 - \beta^2)g(Y, Z)\bar{S}(\xi, U) - (\alpha^2 - \beta^2)\eta(Z)\bar{S}(Y, U) - 2\varepsilon\alpha\beta g(\varphi Y, Z)\bar{S}(\xi, U) \\ & + 2\alpha\beta\eta(Z)\bar{S}(\varphi Y, U) + \varepsilon(Z\alpha)\bar{S}(\varphi Y, U) - \varepsilon g(\varphi Y, Z)\bar{S}(grad\alpha, U) \\ & - \varepsilon g(\varphi Y, \varphi Z)\bar{S}(grad\beta, U) + \varepsilon(Z\beta)\bar{S}(Y, U) - \varepsilon(Z\beta)\eta(Y)\bar{S}(\xi, U) \\ & - \beta g(Y, Z)\bar{S}(\xi, U) + \varepsilon\beta\eta(Z)\bar{S}(Y, U) + \alpha g(\varphi Y, Z)\bar{S}(\xi, U) - \varepsilon\alpha\eta(Z)\bar{S}(\varphi Y, U) \\ & + \varepsilon(\alpha^2 - \beta^2)g(Y, U)\bar{S}(\xi, Z) - (\alpha^2 - \beta^2)\eta(U)\bar{S}(Y, Z) - 2\varepsilon\alpha\beta g(\varphi Y, U)\bar{S}(\xi, Z) \\ & + 2\alpha\beta\eta(U)\bar{S}(\varphi Y, Z) + \varepsilon(U\alpha)\bar{S}(\varphi Y, Z) - \varepsilon g(\varphi Y, U)\bar{S}(grad\alpha, Z) \\ & - \varepsilon g(\varphi Y, \varphi U)\bar{S}(grad\beta, Z) + \varepsilon(U\beta)\bar{S}(Y, Z) - \varepsilon(U\beta)\eta(Y)\bar{S}(\xi, Z) \\ & - \beta g(Y, U)\bar{S}(\xi, Z) + \varepsilon\beta\eta(U)\bar{S}(Y, Z) + \alpha g(\varphi Y, U)\bar{S}(\xi, Z) - \varepsilon\alpha\eta(U)\bar{S}(\varphi Y, Z) \\ & = 0. \end{aligned} \quad (7.3)$$

Using equations (2.1) – (2.5), (2.12), (2.13), (4.17) and (4.21) – (4.26) in equation (7.3), we get

$$\begin{aligned} & [(\alpha^2 - \beta^2) - \varepsilon(\xi\beta) - \varepsilon\beta]S(Y, Z) \\ = & [\varepsilon(n-1)(\alpha^2 - \beta^2)^2 - 2\beta(n-1)(\alpha^2 - \beta^2) - 2(n-1)(\alpha^2 - \beta^2)(\xi\beta) \\ & + 2\varepsilon\beta(n-1)(\xi\beta) + \varepsilon(\xi\beta)^2 + (\varphi grad\beta)\alpha + (n-2)(grad\beta)^2 \\ & + \varepsilon\beta^2(n-2) + (2\beta + \varepsilon)(n-2)(\alpha^2 - \beta^2) + \beta(\alpha^2 - \beta^2) \\ & - 2\alpha^2\beta(n-2) - \varepsilon\alpha(\xi\alpha) - (2\varepsilon\beta + 1)(n-2)(\xi\beta) - \varepsilon\beta(\xi\beta) \\ & - \varepsilon\beta(2\beta + \varepsilon)(n-2) + \varepsilon\alpha^2(n-2)]g(Y, Z) + [-\varepsilon(\varphi grad\beta)\alpha \\ & - \varepsilon(n-2)(grad\beta)^2 - (n-2)(1 + \varepsilon\beta)(\alpha^2 - \beta^2) \\ & + 2\varepsilon\alpha^2\beta(n-2) + \alpha(n-2)(\xi\alpha) + (\varepsilon + \beta)(n-2)(\xi\beta) \\ & + \beta(\varepsilon + \beta)(n-2) - \alpha^2(n-2)]\eta(Y)\eta(Z) + [-2\varepsilon\alpha\beta(n-1)(\alpha^2 - \beta^2) \\ & + 2(n-2)\alpha\beta^2 + 2\alpha\beta(n-1)(\xi\beta) - (n-1)(\alpha^2 - \beta^2)(\xi\alpha) \\ & + \varepsilon\beta(n-2)(\xi\alpha) + \varepsilon(\xi\alpha)(\xi\beta) + (\varphi grad\alpha)\alpha + (n-2) \\ & g(grad\alpha, grad\beta) + \alpha(\alpha^2 - \beta^2) - \varepsilon\alpha(\xi\beta) - 2\alpha\beta(n-2)(\varepsilon + 2\beta) \\ & - (1 + 2\varepsilon\beta)(n-2)(\varepsilon\alpha) + \varepsilon\alpha(\varepsilon + 2\beta)(n-2)]g(\varphi Y, Z) \\ & + [\varepsilon(\xi\alpha) + 2\alpha\beta - \varepsilon\alpha]S(\varphi Y, Z) + [(n-2)(\xi\beta)(Z\beta) \\ & - \varepsilon(\alpha^2 - \beta^2)(\varphi Z)\alpha - \varepsilon(n-2)(\alpha^2 - \beta^2)(Z\beta) + (\xi\beta)(\varphi Z)\alpha \\ & + \beta(\varphi Z)\alpha + \beta(n-2)(Z\beta)]\eta(Y) + [\varepsilon(\alpha^2 - \beta^2)(\varphi Y)\alpha \\ & + \varepsilon(n-2)(\alpha^2 - \beta^2)(Y\beta) - 2\varepsilon\alpha\beta(\varphi^2 Y)\alpha - 2\varepsilon\alpha\beta(n-2)(\varphi Y\beta) \\ & - \beta(\varphi Y)\alpha - \beta(n-2)(Y\beta) + \alpha(\varphi^2 Y)\alpha + \alpha(n-2)(\varphi Y\beta)]\eta(Z) \\ & - (Z\alpha)(\varphi^2 Y)\alpha - (n-2)(Z\alpha)(\varphi Y\beta) - (Z\beta)(\varphi Y)\alpha \\ & - (n-2)(Y\beta)(Z\beta) + (n-2)(Z\beta)(\xi\beta). \end{aligned} \quad (7.4)$$

If  $\alpha = 0$  and  $\beta = \text{constant}$  in equation (6.7), we get

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z), \tag{7.5}$$

where  $a = -\left[\frac{(n-1)\epsilon\beta^4 + (n-2)(grad\beta)^2 + (n-2)\epsilon\beta^2 - (n-2)(\epsilon+2\beta)\beta^2 - (n-2)\epsilon\beta(\epsilon+2\beta) + (2n-3)\beta^3}{\beta(\epsilon+\beta)}\right]$   
 and  $b = -\left[\frac{(n-2)(1+\epsilon\beta)\beta^2 + (n-2)(\epsilon+\beta)\beta - (n-2)\epsilon(grad\beta)^2}{\beta(\epsilon+\beta)}\right]$ .

This shows that  $M^n$  is an  $\eta$ -Einstein manifold. Thus, we can state the following:

**Theorem 7.1.** If an indefinite trans-Sasakian manifold with a semi symmetric metric connection  $\bar{\nabla}$  satisfies  $\bar{R}.\bar{S} = 0$ , then the indefinite trans-Sasakian manifold is an  $\eta$ - Einstein manifold if  $\alpha = 0$  and  $\beta = \text{constant}$ .

### 8 Indefinite trans-Sasakian manifold with respect to a semi-symmetric metric connection satisfies $\bar{P}.\bar{S} = 0$

Now, we consider an indefinite trans-Sasakian manifold with a semi-symmetric connection  $\bar{\nabla}$  satisfying

$$(\bar{P}(X, Y).\bar{S})(Z, U) = 0 \tag{8.1}$$

where  $\bar{P}$  is the projective curvature tensor and  $\bar{S}$  is the Ricci tensor with a semi-symmetric metric connection . Then, we have

$$\bar{S}(\bar{P}(X, Y)Z, U) + \bar{S}(Z, \bar{P}(X, Y)U) = 0 \tag{8.2}$$

Replace  $X = \xi$  in the equation (8.2), we get

$$\bar{S}(\bar{P}(\xi, Y)Z, U) + \bar{S}(Z, \bar{P}(\xi, Y)U) = 0 \tag{8.3}$$

Putting  $X = \xi$  in equation (5.1), we get

$$\bar{P}(\xi, Y)Z = \bar{R}(\xi, Y)Z - \frac{1}{(n-1)}[\bar{S}(Y, Z)\xi - \bar{S}(\xi, Z)Y] \tag{8.4}$$

Using equations (2.1), (2.2), (2.4), (2.11), (4.2), (4.17), (4.23) and (8.4) in equation (8.3), we get



$$\begin{aligned}
& \frac{\varepsilon(n-1)(\alpha^2 - \beta^2) + (n-2)(\varepsilon + \beta)}{(n-1)}g(Y, Z)\bar{S}(\xi, U) - \frac{1}{(n-1)}S(Y, Z)\bar{S}(\xi, U) \quad (8.5) \\
& - \frac{(n-2)}{(n-1)}(1 + \varepsilon\beta)\eta(Y)\eta(Z)\bar{S}(\xi, U) + \frac{\alpha - 2(n-1)\varepsilon\alpha\beta}{(n-1)}g(\varphi Y, Z)\bar{S}(\xi, U) \\
& - \varepsilon g(\varphi Y, Z)\bar{S}(\text{grad}\alpha, U) - \varepsilon g(\varphi Y, \varphi Z)\bar{S}(\text{grad}\beta, U) + 2\alpha\beta\eta(Z)\bar{S}(\varphi Y, U) \\
& + \varepsilon(Z\alpha)\bar{S}(\varphi Y, U) + \varepsilon(Z\beta)\bar{S}(Y, U) - \varepsilon(Z\beta)\eta(Y)\bar{S}(\xi, U) - \varepsilon\alpha\eta(Z)\bar{S}(\varphi Y, U) \\
& - \frac{1}{(n-1)}\varepsilon(\xi\beta)\eta(Z)\bar{S}(Y, U) - \frac{(n-2)}{(n-1)}\varepsilon(Z\beta)\bar{S}(Y, U) - \frac{1}{(n-1)}\varepsilon(\varphi Z)\alpha\bar{S}(Y, U) \\
& + \frac{\varepsilon(n-1)(\alpha^2 - \beta^2) + (n-2)(\varepsilon + \beta)}{(n-1)}g(Y, U)\bar{S}(\xi, Z) - \frac{1}{(n-1)}S(Y, U)\bar{S}(\xi, Z) \\
& - \frac{(n-2)}{(n-1)}(1 + \varepsilon\beta)\eta(Y)\eta(U)\bar{S}(\xi, Z) + \frac{\alpha - 2(n-1)\varepsilon\alpha\beta}{(n-1)}g(\varphi Y, U)\bar{S}(\xi, Z) \\
& - \varepsilon g(\varphi Y, U)\bar{S}(\text{grad}\alpha, Z) - \varepsilon g(\varphi Y, \varphi U)\bar{S}(\text{grad}\beta, Z) + 2\alpha\beta\eta(U)\bar{S}(\varphi Y, Z) \\
& + \varepsilon(U\alpha)\bar{S}(\varphi Y, Z) + \varepsilon(U\beta)\bar{S}(Y, Z) - \varepsilon(U\beta)\eta(Y)\bar{S}(\xi, Z) - \varepsilon\alpha\eta(U)\bar{S}(\varphi Y, Z) \\
& - \frac{1}{(n-1)}\varepsilon(\xi\beta)\eta(U)\bar{S}(Y, Z) - \frac{(n-2)}{(n-1)}\varepsilon(U\beta)\bar{S}(Y, Z) - \frac{1}{(n-1)}\varepsilon(\varphi U)\alpha\bar{S}(Y, Z) \\
& = 0.
\end{aligned}$$

Putting  $U = \xi$  and using equations (2.1) – (2.5), (4.17) and (4.21) – (4.26) in equation (8.5), we get

$$\begin{aligned}
& [(\alpha^2 - \beta^2) - \varepsilon(\xi\beta) - \varepsilon\beta]S(Y, Z) \quad (8.6) \\
& = [\varepsilon(n-1)(\alpha^2 - \beta^2)^2 + (n-2)(\varepsilon + \beta)(\alpha^2 - \beta^2) - \beta(n-1)(\alpha^2 - \beta^2) \\
& - \beta(n-2)(1 + \varepsilon\beta) - 2(n-1)(\xi\beta)(\alpha^2 - \beta^2) - (n-2)(1 + \varepsilon\beta)(\xi\beta) \\
& - 2\alpha^2\beta(n-2) - \varepsilon\alpha(n-2)(\xi\alpha) + \varepsilon\alpha^2(n-2) + \varepsilon\beta(n-1) + \varepsilon(\xi\beta)^2 \\
& + (\varphi\text{grad}\alpha)\alpha + (n-2)(\text{grad}\beta)^2]g(Y, Z) + [(n-2)\beta(\varepsilon + \beta) - (n-2)(\alpha^2 - \beta^2) \\
& + 2(n-2)\varepsilon\alpha^2\beta + \alpha(n-2)(\xi\alpha) + (n-2)(\varepsilon + \beta)(\xi\beta) - \alpha^2(n-2) \\
& - \varepsilon(\varphi\text{grad}\beta)\alpha - \varepsilon(n-2)(\text{grad}\beta)^2]\eta(Y)\eta(Z) + [\alpha(\alpha^2 - \beta^2) \\
& - 2(n-1)\varepsilon\alpha\beta(\alpha^2 - \beta^2) - 2n\alpha\beta^2 - \varepsilon\alpha(\xi\beta) - \varepsilon\beta(\xi\alpha) + 2\alpha\beta(\xi\beta) \\
& - 2(n-2)\alpha\beta(2\beta + \varepsilon) - (n-2)(1 + 2\varepsilon\beta)(\xi\alpha) + \alpha(n-2)(2\varepsilon\beta + 1) \\
& - (n-1)(\alpha^2 - \beta^2)(\xi\alpha) + (n-1)\varepsilon\beta(\xi\alpha) + \varepsilon(\xi\alpha)(\xi\beta) + (\varphi\text{grad}\alpha)\alpha \\
& + (n-2)g(\text{grad}\alpha, \text{grad}\beta)]g(\varphi Y, Z) + [2\alpha\beta + \varepsilon(\xi\alpha) - \varepsilon\alpha]S(\varphi Y, Z) \\
& + [\varepsilon(n+3)(\alpha^2 - \beta^2)(Z\beta) + \beta(n-2)(Z\beta) - \varepsilon(\alpha^2 - \beta^2)(\varphi Z)\alpha \\
& + (n-1)\beta(\varphi Z)\alpha + (\xi\beta)(\varphi Z)\alpha]\eta(Y) + [-2\varepsilon\alpha\beta(\varphi^2 Y)\alpha - 2\varepsilon\alpha\beta(n-2)(\varphi Y\beta) \\
& + \alpha(\varphi^2 Y)\alpha + \alpha(n-2)(\varphi Y\beta) + \varepsilon(\alpha^2 - \beta^2)(\varphi Y)\alpha + \varepsilon(n-2)(\alpha^2 - \beta^2)(Y\beta) \\
& - \beta(\varphi Y)\alpha - \beta(n-2)(Y\beta)]\eta(Z) - (Z\alpha)(\varphi^2 Y)\alpha - (n-2)(Z\beta)(\varphi Y\beta) \\
& - (Z\beta)(\varphi Y)\alpha - \beta(n-2)(Y\beta).
\end{aligned}$$

If  $\alpha = 0$  and  $\beta = \text{constant}$  in equation (8.6), we get

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z), \quad (8.7)$$

where  $a = -\left[\frac{(n-1)\varepsilon\beta^4 - (n-2)\beta^2(\varepsilon+\beta) + (n-1)\beta^3 - (n-2)\beta(1+\varepsilon\beta + (n-1)\varepsilon\beta + (n-2)(\text{grad}\beta)^2)}{\beta(\varepsilon+\beta)}\right]$

and  $b = -\left[\frac{(n-2)\beta(\varepsilon+\beta) + (n-2)\beta^2 - (n-2)\varepsilon(\text{grad}\beta)^2}{\beta(\varepsilon+\beta)}\right]$ .

This result shows that the manifold under the consideration is an  $\eta$ -Einstein manifold. Thus, we can state the following theorem:

**Theorem 8.1.** If an indefinite trans-Sasakian manifold with a semi symmetric metric connection  $\bar{\nabla}$  satisfies  $\bar{P}.\bar{S} = 0$ , then the indefinite trans-Sasakian manifold is an  $\eta$ -Einstein manifold if  $\alpha = 0$  and  $\beta = \text{constant}$ .

## 9 Weyl conformal curvature tensor on indefinite trans-Sasakian manifold with a semi-symmetric metric connection

The weyl conformal curvature tensor  $\bar{C}$  of type (1, 3) of  $M^n$  an  $n$ -dimensional indefinite trans-Sasakian manifold with semi-symmetric metric connection  $\bar{\nabla}$  is given by [16]

$$\begin{aligned} \bar{C}(X, Y)Z &= \bar{R}(X, Y)Z - \frac{1}{(n-2)}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y + g(Y, Z)\bar{Q}X \\ &\quad - g(X, Z)\bar{Q}Y] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (9.1)$$

where  $\bar{Q}$  is the Ricci operator with respect to the semi-symmetric metric connection  $\bar{\nabla}$ . Let  $M^n$  be an  $n$ -dimensional indefinite trans-Sasakian manifold. The Weyl conformal curvature tensor  $\bar{C}$  of  $M^n$  with respect to the semi-symmetric metric connection  $\bar{\nabla}$  is defined in equation (9.1).

Now, taking inner product with  $U$  in equation (9.1), we get

$$\begin{aligned} &g(\bar{C}(X, Y)Z, U) \\ &= g(\bar{R}(X, Y)Z, U) - \frac{1}{(n-2)}[\bar{S}(Y, Z)g(X, U) - \bar{S}(X, Z)g(Y, U) \\ &\quad + g(Y, Z)g(\bar{Q}X, U) - g(X, Z)g(\bar{Q}Y, U)] + \frac{\bar{r}}{(n-1)(n-2)} \\ &\quad [g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]. \end{aligned} \quad (9.2)$$

Using equations (2.4), (4.2), (4.17), (4.18) and (4.20) in equation (9.2), we get

$$\begin{aligned} &\bar{C}(X, Y, Z, U) \\ &= g(R(X, Y)Z, U) - \frac{1}{(n-2)}[S(Y, Z)g(X, U) - S(X, Z)g(Y, U) \\ &\quad + g(Y, Z)g(QX, U) - g(X, Z)g(QY, U)] + \frac{r}{(n-1)(n-2)} \\ &\quad [g(Y, Z)g(X, U) - g(X, Z)g(Y, U)] \end{aligned} \quad (9.3)$$

where  $g(\bar{C}(X, Y)Z, U) = \bar{C}(X, Y, Z, U)$  and  $g(R(X, Y)Z, U) = C(X, Y, Z, U)$  are Weyl curvature tensor with respect to semi-symmetric metric connection and metric connection respectively. we have

$$\bar{C}(X, Y, Z, U) = C(X, Y, Z, U), \tag{9.4}$$

where

$$\begin{aligned} & C(X, Y, Z, U) \tag{9.5} \\ = & g(R(X, Y)Z, U) - \frac{1}{(n-2)}[S(Y, Z)g(X, U) - S(X, Z)g(Y, Z) \\ & + g(Y, Z)g(QX, U) - g(X, Z)g(QY, U)] + \frac{r}{(n-1)(n-2)} \\ & [g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]. \end{aligned}$$

**Theorem 9.1.** The Weyl conformal curvature tensor of an indefinite trans-Sasakian manifold  $M^n$  with respect to a metric connection is equal to the weyl conformal curvature of with respect to semi-symmetric metric connection.

### 10 Indefinite trans-Sasakian manifold with weyl conformal flat conditions with a semi-symmetric metric connection

Let us consider that the indefinite trans-Sasakian manifold  $M^n$  with respect to the semi-symmetric metric connection is Weyl conformally flat, that is  $\bar{C} = 0$ . Then from equation (9.1), we get

$$\begin{aligned} \bar{R}(X, Y)Z &= \frac{1}{(n-2)}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y + g(Y, Z)\bar{Q}X \\ & - g(X, Z)\bar{Q}Y] - \frac{\bar{r}}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y]. \end{aligned} \tag{10.1}$$

Now, taking the inner product of equation (10.1) with  $U$ . Then, we get

$$\begin{aligned} g(\bar{R}(X, Y)Z, U) &= \frac{1}{(n-2)}[\bar{S}(Y, Z)g(X, U) - \bar{S}(X, Z)g(Y, U) \\ & + g(Y, Z)g(\bar{Q}X, U) - g(X, Z)g(\bar{Q}Y, U)] \\ & - \frac{\bar{r}}{(n-1)(n-2)}[g(Y, Z)g(X, U) \\ & - g(X, Z)g(Y, U)]. \end{aligned} \tag{10.2}$$

Using equations (2.4), (4.2), (4.17), (4.18) and (4.20) in equation (10.2), we get

$$\begin{aligned} g(R(X, Y)Z, U) &= \frac{1}{(n-2)}[S(Y, Z)g(X, U) - S(X, Z)g(Y, U) \\ & + g(Y, Z)g(QX, U) - g(X, Z)g(QY, U)] \\ & - \frac{r}{(n-1)(n-2)}[g(Y, Z)g(X, U) \\ & - g(X, Z)g(Y, U)]. \end{aligned} \tag{10.3}$$

Putting  $X = U = \xi$  in equation (10.3) and using equations (2.2), (2.3) and (2.4), we get

$$\begin{aligned}
 &g(R(\xi, Y)Z, \xi) \tag{10.4} \\
 = &\frac{1}{(n-2)}[\varepsilon S(Y, Z) - \varepsilon\eta(Y)S(\xi, Z) + g(Y, Z)S(\xi, \xi) \\
 &- \varepsilon\eta(Z)S(Y, \xi)] - \frac{r}{(n-1)(n-2)}[\varepsilon g(Y, Z) - \eta(Y)\eta(Z)],
 \end{aligned}$$

Where  $g(QY, Z) = S(Y, Z)$ .

Now, using equations (2.12), (2.13) and (2.16), we get

$$\begin{aligned}
 S(Y, Z) = & [(\xi\beta) - \varepsilon(\alpha^2 - \beta^2) + \frac{r}{(n-1)}]g(Y, Z) + [\varepsilon(n-4)(\xi\beta) \tag{10.5} \\
 & + n(\alpha^2 - \beta^2) - \frac{\varepsilon r}{(n-1)}]\eta(Y)\eta(Z) - [2\varepsilon(n-2)\alpha\beta \\
 & + (n-2)(\xi\alpha)]g(\varphi Y, Z) - [\varepsilon(\varphi Z)\alpha + \varepsilon(Z\beta)(n-2)]\eta(Y) \\
 & - [\varepsilon(\varphi Y)\alpha + \varepsilon(n-2)(Y\beta)]\eta(Z).
 \end{aligned}$$

If  $\alpha = 0$  and  $\beta = \text{constant}$  in equation (8.6), we get

$$S(Y, Z) = [\varepsilon\beta^2 + \frac{r}{(n-1)}]g(Y, Z) + [-n\beta^2 - \frac{\varepsilon r}{(n-1)}]\eta(Y)\eta(Z). \tag{10.6}$$

Therefore

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z)$$

where  $a = [\varepsilon\beta^2 + \frac{r}{(n-1)}]$  and  $b = [-n\beta^2 - \frac{\varepsilon r}{(n-1)}]$ .

This shows that  $M^n$  is an  $\eta$ -Einstein manifold. Thus, we can state as follows:

**Theorem 10.1.** Let  $M^n$  be a  $n$ -dimensional Weyl conformally flat indefinite trans-Sasakian manifold with respect to semi symmetric metric connection  $\tilde{\nabla}$  is an  $\eta$ -Einstein manifold if  $\alpha = 0$  and  $\beta = \text{constant}$ .

Now, taking equation (9.1), we have

$$\begin{aligned}
 \tilde{C}(X, Y)Z = & \tilde{R}(X, Y)Z - \frac{1}{(n-2)}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y \tag{10.7} \\
 & + g(Y, Z)\tilde{Q}X - g(X, Z)\tilde{Q}Y] \\
 & + \frac{\tilde{r}}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y].
 \end{aligned}$$

Using equations (2.15), (4.2)(4.17), (4.18) and (4.20) in equation (10.7), we get

$$\begin{aligned}
 & \bar{C}(X, Y)Z \\
 = & C(X, Y)Z + (2\beta + \varepsilon)[g(X, Z)Y - g(Y, Z)X ] \\
 & + (\varepsilon + \beta)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\xi \\
 & + (1 + \varepsilon\beta)\eta(Z)[\eta(Y)X - \eta(X)Y ] - \alpha[g(\varphi X, Z)Y \\
 & - g(\varphi Y, Z)X - g(Y, Z)\varphi X + g(X, Z)\varphi Y] - \frac{1}{(n - 2)} \\
 & [(1 + \varepsilon\beta)(n - 2)\eta(Y)\eta(Z) - ((2\beta + \varepsilon)(n - 2) + \beta)g(Y, Z)X \\
 & + \alpha(n - 2)g(\varphi Y, Z)X + ((2\beta + \varepsilon)(n - 2) + \beta)g(X, Z)Y \\
 & - (1 + \varepsilon\beta)(n - 2)\eta(X)\eta(Z)Y - \alpha(n - 2)g(\varphi X, Z)Y \\
 & - ((2\beta + \varepsilon)(n - 2) + \beta)g(Y, Z)X + (\varepsilon + \beta)(n - 2)g(Y, Z)\eta(X)\xi \\
 & + \alpha(n - 2)g(Y, Z)\varphi X + ((2\beta + \varepsilon)(n - 2) + \beta)g(X, Z)Y \\
 & - (\varepsilon + \beta)(n - 2)g(X, Z)\eta(Y)\xi - \alpha(n - 2)g(X, Z)\varphi Y] \\
 & - \frac{2\beta + (2\beta + \varepsilon)(n - 2)}{(n - 2)}[g(Y, Z)X - g(X, Z)Y].
 \end{aligned}
 \tag{10.8}$$

Let  $X$  and  $Y$  are orthogonal basis to  $\xi$  . Putting  $Z = \xi$  and using equations (2.1), (2.2) and (2.4) in equation (10.8), we get

$$\bar{C}(X, Y)\xi = C(X, Y)\xi.
 \tag{10.9}$$

**Theorem 10.2.** A  $n$ -dimensional indefinite trans-Sasakian manifold  $M^n$  is Weyl  $\xi$ -conformally flat with respect to the semi-symmetric metric connection if and only if the manifold is also weyl  $\xi$ -conformally flat with respect to the metric connection provided that the vector fields are horizontal vector fields.

**Example 10.3.** Let us consider a 3-dimensional manifold  $M = \{(x, y, z) \in R^3 : z \neq 0\}$ , where  $(x, y, z)$  are the standard co-ordinates in  $R^3$ . we choose the vector fields

$$e_1 = -z\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \quad e_2 = -z\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

which are linearly independent at each point of  $M$ . Let  $g$  be the Riemannian metric defined by

$$\begin{aligned}
 g(e_1, e_2) &= g(e_2, e_3) = g(e_3, e_1) = 0, \\
 g(e_1, e_1) &= g(e_2, e_2) = g(e_3, e_3) = \varepsilon.
 \end{aligned}$$

Let  $\eta$  be the 1-form defined by  $\eta(X) = \varepsilon g(X, \xi)$  for any  $X \in \chi(M)$  and  $\varphi$  be the (1, 1) tensor field defined by

$$\varphi(e_1) = e_2, \quad \varphi(e_2) = -e_1, \quad \varphi(e_3) = 0.$$

Then using the linearity of  $\varphi$  and  $g$ , we have

$$\begin{aligned}
 \varphi^2 X &= -X + \eta(X)e_3, \quad \eta(e_3) = 1, \\
 g(\varphi X, \varphi Y) &= g(X, Y) - \varepsilon\eta(X)\eta(Y),
 \end{aligned}$$

where  $e_3 = \xi$ , the structure  $(\varphi, \xi, \eta, g)$  defines an indefinite Trans-Sasakian structure on  $M$ . Let  $\nabla$  the Levi-Civita connection with metric  $g$ , then we have

$$[e_1, e_2] = \varepsilon(-ye_2 - z^2e_3), [e_1, e_3] = \frac{1}{z}\varepsilon e_1, [e_2, e_3] = \frac{1}{z}\varepsilon e_2.$$

The Riemannian connection  $\nabla$  of the metric  $g$  is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y),$$

which known as Koszul's formula Taking  $e_3 = \xi$  and using Koszul's formula, we have

$$\begin{aligned} \nabla_{e_1} e_1 &= -\frac{1}{z}\varepsilon e_3, \nabla_{e_1} e_2 = -\frac{1}{2}\varepsilon z^2 e_3, \nabla_{e_1} e_3 = \frac{1}{z}\varepsilon e_1 + \frac{1}{2}\varepsilon z^2 e_2, \\ \nabla_{e_2} e_1 &= \frac{1}{2}\varepsilon z^2 e_3 + \varepsilon y e_2, \nabla_{e_2} e_2 = \varepsilon(-ye_1 - \frac{1}{z}e_3), \nabla_{e_3} e_3 = 0, \\ \nabla_{e_2} e_3 &= \frac{1}{z}\varepsilon e_2 - \frac{1}{2}\varepsilon z^2 e_1, \nabla_{e_3} e_1 = \frac{1}{2}\varepsilon z^2 e_2, \nabla_{e_3} e_2 = -\frac{1}{2}\varepsilon z^2 e_1. \end{aligned}$$

Therefore, the semi-symmetric metric connection on  $M$  is given by

$$\begin{aligned} \bar{\nabla}_{e_1} e_1 &= -\frac{1}{z}\varepsilon e_3 - \varepsilon e_3, \bar{\nabla}_{e_1} e_2 = -\frac{1}{2}\varepsilon z^2 e_3, \bar{\nabla}_{e_3} e_3 = 0 \\ \bar{\nabla}_{e_1} e_3 &= \frac{1}{z}\varepsilon e_1 + \frac{1}{2}\varepsilon z^2 e_2 + \varepsilon e_1, \bar{\nabla}_{e_3} e_2 = -\frac{1}{2}\varepsilon z^2 e_1 \\ \bar{\nabla}_{e_2} e_1 &= \frac{1}{2}\varepsilon z^2 e_3 + \varepsilon y e_2, \bar{\nabla}_{e_2} e_2 = \varepsilon(-ye_1 - \frac{1}{z}e_3 - e_3), \\ \bar{\nabla}_{e_2} e_3 &= \frac{1}{z}\varepsilon e_2 - \frac{1}{2}\varepsilon z^2 e_1 + \varepsilon e_2, \bar{\nabla}_{e_3} e_1 = \frac{1}{2}\varepsilon z^2 e_2. \end{aligned}$$

Now, for  $\xi = e_3$ , above results satisfy

$$\bar{\nabla}_X \xi = -\varepsilon \alpha(\varphi X) + (1 + \varepsilon \beta)(X - \eta(X)\xi),$$

with  $\alpha = -\frac{1}{2}z^2$  and  $\beta = \frac{1}{z}$ . Consequently  $M(\varphi, \xi, \eta, g, \varepsilon)$  is a indefinite trans Sasakian manifold with semi-symmetric connection.

**Acknowledgement.** Sushil Kumar is thankful to UGC, New Dehli (**India**) for a fellowship.

## References

- [1] C. S. Bagewadi, *On totally real submanifolds of a Kahlerian manifold admitting Semi symmetric metric F-connection*. Indian. J. Pure. Appl. Math, **13**, 528-536, (1982) .
- [2] C. S. Bagewadi and N. B. Gatti, *On irrotational quasi-conformal curvature tensor*. Tensor.N.S., **64**, 284-258, (2003).

- [3] C. S. Bagewadi and E. Girish Kumar, *Note on Trans-Sasakian Manifolds*. *Tensor*. N. S., **65**, 80-88 (2004).
- [4] A. Bejancu and K. L. Duggal, *Real hypersurfaces of indefinite Kahler manifolds*. *Int. J. Math. Math. sci.*, **16**, 545-556, (1993). .
- [5] D. E. Blair, *Contact manifolds in Riemannian geometry*. *Lecture note in Mathematics*, 509, Springer-Verlag Berlin-New York, 1976.
- [6] U. C. De. and A. Sarkar, *On  $(\varepsilon)$ -Kenmotsu manifolds*. *Hadronic journal*, **32** , 231-242, (2009).
- [7] U. C. De and Absos Ali Shaikh, *K-contact and Sasakian manifolds with conservative quasi-conformal curvature tensor*. *Bull. Cal. Math. Soc.*, **89**, 349-354, (1997).
- [8] A. Friedmann and J. A. Schouten, *Über die geometrie der holbsymmetrischen Übertragungurgen*. *Math. Zeitschr.* **21**, 211-233, (1924).
- [9] H. A. Hayden, *Subspaces of space with torsion*. *Proc. Lond. Math. Soc.* **34** , 27-50, (1932) .
- [10] S. I. Hussain and A. Sharafuddin, *Semi-symmetric metric connections in almost contact manifolds*. *Tensor*, N. S.,**30**, 133-139, (1976).
- [11] Amur Kumar and S. S. Pujar, *On Submanifolds of a Riemannian manifold admitting a metric semi-symmetric connection*. *Tensor*, N. S., **32**, 35-38, (1978).
- [12] R. Kumar, R. Rani and R. K. Nagaich, *On sectional curvature of  $(\varepsilon)$ -Sasakian manifolds*. *Int. J. Math. Math. sci.*, (2007), ID. 93562.
- [13] J. C. Marrero, *The local structures of trans-Sasakian manifolds*. *Ann. Mat. Pura. Appl*, (4), **162**, 77-86, (1992).
- [14] Halammanavar G. Nagaraja, Rangaswami C. Premalatha and Ganganna Somashekara, *On an  $(\varepsilon, \delta)$  trans-Sasakian structure*. *Proceedins of the Estonian Academy of Sciences*, (1), **61**, 20-28 (2012).
- [15] J. A. Oubiñá, *New classes of almost contact metric structures*. *Publicationes Mathematicae Debrecen* ,vol.**32**, 187-193, (1985).
- [16] M. M. Tripathi, *Ricci solitons in contact metric manifolds*. *arXiv:0801.4222v1,[math.DG]*,28, (2008).
- [17] K. Yano, *On semi-symmetric metric connections*. *Revue Roumaine de Math. Pures et Appliques.*, **15** 1579-1586, (1970) .
- [18] X.Xufeng and C. Xiaoli, *Two theorems on  $(\varepsilon)$ -Sasakian manifolds*. *Int. J. Math. Sci.*, **21** 249 -254, (1998).